ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, Nos. 3–4, 1976

ON A CLASS OF CONVEX POLYTOPES

BY

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ABSTRACT

Let $\mathcal{P}$ denote the class of convex polytopes P having the following property: If Q_1 and Q_2 are any subpolytopes of P with no vertex in common, then $Q_1 \cap Q_2$ is either empty or a single point. A characterization of $\mathcal{P}$ is given which implies the characterization of strongly positively independent sets due to McKinney, Hansen and Klee.

1. Introduction

The purpose of this note is to prove a theorem on a class of convex polytopes which is analogous to the characterization of strongly positively independent sets given by McKinney [5] and Hansen and Klee [4].

Throughout P will denote a polytope, that is, a convex polytope in some d-dimensional affine space R^d. If P is d-dimensional, it will be called a d-polytope. For general information about polytopes we refer to Grünbaum [3]. The standard abbreviations conv, aff, relint, dim, card will be used for convex hull, affine hull, relative interior, dimension, cardinality. The vertex set of P will be denoted by vert P.

A polytope Q is called a *subpolytope* of P if vert $Q \subset$ vert P. We write $\mathcal{P}$ for the class of polytopes P with the following property: If Q_1 and Q_2 are subpolytopes of P such that vert $Q_1 \cap$ vert $Q_2 = \emptyset$, then $Q_1 \cap Q_2$ is either empty or a single point. We shall give a simple geometrical description of the class $\mathcal{P}$. Clearly, if $P \in \mathcal{P}$ and Q is a subpolytope of P, then $Q \in \mathcal{P}$. Every simplex belongs to $\mathcal{P}$, and so does every d-polytope with $d+2$ vertices, as can be deduced from the well-known structure of such polytopes. See [3, p. 98].

In order to formulate our result we introduce the concept of a *cross* of polytopes. By this we mean the following. Let $P, Q_1, \cdots, Q_k$ be polytopes such that $P = \operatorname{conv}(Q_1 \cup \cdots \cup Q_k)$, $\dim P = \dim Q_1 + \cdots + \dim Q_k$, and

Received June 30, 1975

$\operatorname{relint} Q_1 \cap \cdots \cap \operatorname{relint} Q_k$ is a single point. Then P is called the cross of $Q_1, \cdots, Q_k$, and we write

$$P = Q_1 \perp \cdots \perp Q_k.$$

Our theorem is as follows.

THEOREM. *Let P be a polytope in R^d. Then $P \in \mathcal{P}$ if and only if, for some $r \geqq 0$ and simplices $S_1, \cdots, S_k$ in R^d, P is an r-fold pyramid with basis $S_1 \perp \cdots \perp S_k$.*

For the definition of an r-fold pyramid, see [3, p. 55].

Let $P \in \mathcal{P}$ be a polytope in R^d as described in the theorem. We define the *center* z of P to be the point $\operatorname{relint} S_1 \cap \cdots \cap \operatorname{relint} S_k$, if $k > 1$, and an arbitrary point of $\operatorname{relint} P$, if $k = 1$. The theorem implies that if Q_1 and Q_2 are any subpolytopes of P satisfying $\operatorname{vert} Q_1 \cap \operatorname{vert} Q_2 = \emptyset$, then $Q_1 \cap Q_2 \subset \{z\}$.

Another consequence of the theorem is that no d-polytope $P \in \mathcal{P}$ can have more than $2d$ vertices. If the number of vertices is exactly $2d$, then P is a cross of d segments, that is, P is projectively equivalent to the regular d-crosspolytope. By a *cross set* in R^d we understand the vertex set of a cross of d segments together with its center.

The following is an easy deduction from the theorem.

COROLLARY. *Let X be a set in R^d. If* $\operatorname{card} X > 2d + 1$, *then X contains two disjoint subsets whose convex hulls have at least two points in common. The same is true when* $\operatorname{card} X = 2d + 1$ *unless X is a cross set in R^d.*

2. Proof of the theorem

Before proceeding with the proof we recall that two sets $X_1, X_2 \subset R^d$ are said to be *separated* by a hyperplane H if they lie one in each of the two closed half-spaces bounded by H. If X_1 and X_2 are convex and $\operatorname{aff}(X_1 \cup X_2) = R^d$, then X_1 and X_2 may be separated by a hyperplane if and only if $\operatorname{relint} X_1 \cap \operatorname{relint} X_2 = \emptyset$. See [3, p. 11].

The "if" part of the theorem is easily verified. For the "only if" part we use induction on d and the number of vertices of P. Let $P \in \mathcal{P}$ denote a d-polytope in R^d. We assume that P is not a simplex because otherwise there is nothing to prove. Moreover, since a pyramid belongs to $\mathcal{P}$ if and only if its basis does, we may also assume that P is non-pyramidal. It is to be shown that P is a cross of at least two simplices.

Let us choose any vertex $v \in \operatorname{vert} P$ and let Q denote the subpolytope of P spanned by $\operatorname{vert} P \setminus \{v\}$. Then $Q \in \mathcal{P}$ and $\dim Q = d$. The inductive hypothesis implies that Q is either a pyramid or a cross of at least two simplices.

The second case cannot arise. Indeed, assume that $Q = S_1 \perp \cdots \perp S_k$, where $k > 1$, and let z be the center of Q. Since $v \neq z$ we have, without loss of generality, $v \notin \operatorname{aff} S_1$. Let $T_1 = \operatorname{conv}(S_1 \cup \{v\})$ and $T_2 = S_2 \perp \cdots \perp S_k$. Then $\operatorname{vert} T_1 \cap \operatorname{vert} T_2 = \varnothing$. Since $\operatorname{aff} Q = R^d$ and $\operatorname{relint} S_1 \cap \operatorname{relint} T_2 \neq \varnothing$ it follows that S_1 and T_2 cannot be separated by a hyperplane. The same is true for T_1 and T_2, whence $\operatorname{relint} T_1 \cap \operatorname{relint} T_2 \neq \varnothing$. But T_1 is a pyramid with basis S_1. Therefore $z \notin \operatorname{relint} T_1$. We deduce that T_1 and T_2 intersect in at least two points, contrary to the hypothesis $P \in \mathcal{P}$.

Hence Q is a pyramid. We first observe that if Q is a simplex, then P is a simplicial d-polytope with $d+2$ vertices, and therefore P is a cross of two simplices. See [3, p. 98].

It remains to consider the case in which Q is an r-fold pyramid with basis $S_1 \perp \cdots \perp S_k$ for some $r \geqq 1$ and $k \geqq 2$. Let z be the center of Q and let $v_1, \cdots, v_r$ denote the apexes of the r pyramids which make up Q. Further, let $V = \{v, v_1, \cdots, v_r\}$ and $R = \operatorname{aff}(S_1 \perp \cdots \perp S_k)$.

We claim that every open half-space bounded by a hyperplane through R contains points of V. In fact, assume that this were not true. Since P is non-pyramidal, at most $r-1$ points of V can lie on a hyperplane through R. In particular, $V \cap R = \varnothing$. Now R has codimension r and so there exists, by Radon's theorem [3, p. 16] in R^{r-1}, a partition of V into non-empty subsets V_1 and V_2 which cannot be separated by any hyperplane through R. Let $T_1 = \operatorname{conv}(S_1 \cup V_1)$ and $T_2 = \operatorname{conv}((S_2 \perp \cdots \perp S_k) \cup V_2)$. Then T_1 and T_2 are subpolytopes of P with no vertex in common. Since $\operatorname{relint} S_1 \cap \operatorname{relint}(S_2 \perp \cdots \perp S_k) \neq \varnothing$, every hyperplane which separates S_1 and $S_2 \perp \cdots \perp S_k$ must contain R and so does not separate V_1 and V_2. Hence T_1 and T_2 cannot be separated by a hyperplane. This yields $\operatorname{relint} T_1 \cap \operatorname{relint} T_2 \neq \varnothing$. On the other hand, we have $z \in T_1 \cap T_2$ and $z \notin \operatorname{relint} T_1 \cup \operatorname{relint} T_2$. Thus $T_1 \cap T_2$ contains at least two points, contrary to hypothesis.

We deduce that V is the vertex set of an r-dimensional simplex S_{k+1} which intersects R in a single point $y \in \operatorname{relint} S_{k+1}$. Assume that $y \neq z$. Then, without loss of generality, $y \notin \operatorname{aff} S_1$, and with reasoning similar to that used above we find that $\operatorname{conv}(S_1 \cup \{y\})$ and $S_2 \perp \cdots \perp S_k$ have at least two points in common. As the same is true of $\operatorname{conv}(S_1 \cup S_{k+1})$ and $S_2 \perp \cdots \perp S_k$, the hypothesis $P \in \mathcal{P}$ is contradicted. Hence $y = z$, and consequently $P = S_1 \perp \cdots \perp S_k \perp S_{k+1}$. This completes the proof of the theorem.

3. Remarks

1) Since positive sets are convex, the theorem implies the characterization of strongly positively independent sets proved by Hansen and Klee [4] and earlier by McKinney [5]. A set $X \subset R^d \setminus \{o\}$ is called *strongly positively independent* provided that $\operatorname{pos} X_1 \cap \operatorname{pos} X_2 \subset \{o\}$ whenever X_1 and X_2 are disjoint subsets of X. Here o is the origin of R^d and $\operatorname{pos} X$ denotes the positive hull of X. It turns out that, in our notation, X is strongly positively independent if and only if $X = \operatorname{vert} P$ for some polytope $P \in \mathcal{P}$ whose center coincides with o. Although our theorem and the Hansen-Klee result are closely related we have been unable to derive the former from the latter. As for the corollary to the theorem, we remark that it implies the well-known intersection theorems for positive sets due to Steinitz and Robinson; see [4].

2) Let $P \in \mathcal{P}$ be a polytope in R^d as described in the theorem. Following Hansen and Klee [4] we define the *invariant* of P to be the sequence $(r; d_1, \cdots, d_k)$, where $d_1, \cdots, d_k$ are the dimensions of $S_1, \cdots, S_k$ arranged in increasing order and with proper multiplicity. Two d-polytopes $P_1, P_2 \in \mathcal{P}$ with centers z_1, z_2 have the same invariant if and only if there exists a non-singular projective transformation Γ of R^d such that $\Gamma P_1 = P_2$ and $\Gamma z_1 = z_2$. The number of equivalence classes under the corresponding equivalence relation is

$$p(1) + \cdots + p(d) - d + 1,$$

where $p(n)$ denotes the number of partitions of n.

3) Let $\mathcal{P}_s$ denote the class of *simplicial* polytopes in $\mathcal{P}$ corresponding to the case $r = 0$ of the theorem. The class $\mathcal{P}_s$ is dual to the class of direct linear sums of simplices investigated by Gruber [1], [2], Schneider [8] and, more recently, by McMullen, Schneider and Shephard [6]. To be precise, let $P \in \mathcal{P}_s$ be a d-polytope in R^d with center o and let P^* be the polar set of P; see [3, p. 47]. Then P^* is a direct linear sum of simplices, P is dual to P^* and the correspondence between P and P^* is one-to-one. The number of equivalence classes of d-polytopes in $\mathcal{P}_s$ is $p(d)$.

4) *A Radon partition* of a finite set $X \subset R^d$ is a pair $\{X_1, X_2\}$ of subsets of X such that $X_1 \cap X_2 = \emptyset$, $X_1 \cup X_2 = X$ and $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset$. Extending the definition of the class $\mathcal{P}$ we write $X \in \mathcal{P}$ provided that $\operatorname{conv} X_1 \cap \operatorname{conv} X_2$ is a single point whenever $\{X_1, X_2\}$ is a Radon partition of X. The theorem implies that $X \in \mathcal{P}$ if and only if, for some polytope $P \in \mathcal{P}$ with center z, either $X = \operatorname{vert} P$ or $X = \operatorname{vert} P \cup \{z\}$. If only sets which affinely span R^d are considered, then the number of equivalence classes in the sense defined above is

$$2p(1)+\cdots+2p(d)-d+1.$$

5) The corollary to the theorem supplements the results of Reay [7] on (r,k)-divisible sets. A set $X \subset R^d$ is said to be *(r,k)-divisible* if it can be partitioned into r pairwise disjoint subsets whose convex hulls intersect in a set of dimension at least k. The corollary shows that if $\operatorname{card} X > 2d+1$ then X is $(2,1)$-divisible. Since a cross set in R^d is not $(2,1)$-divisible, the result is best possible. It would be interesting to prove an analogous theorem for $(r,1)$-divisible sets.

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